JHEP

Published by Institute of Physics Publishing for SISSA/ISAS

Received: *December 19, 2001*
Accepted: *April 16, 2002*

Infinitely many star products to play with

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Abstract: While there has been growing interest for noncommutative spaces in recent times, most examples have been based on the simplest noncommutative algebra: $[x_i, x_j] = i\theta_{ij}$. Here we present new classes of (non-formal) deformed products associated to linear Lie algebras of the kind $[x_i, x_j] = ic_{ij}^k x_k$. For all possible three-dimensional cases, we define a new star product and discuss its properties. To complete the analysis of these novel noncommutative spaces, we introduce *noncompact* spectral triples, and the concept of *star triple*, a specialization of the spectral triple to deformations of the algebra of functions on a noncompact manifold. We examine the generalization to the noncompact case of Connes' conditions for noncommutative spin geometries, and, in the framework of the new star products, we exhibit some candidates for a Dirac operator. On the technical level, properties of the Moyal multiplier algebra $M(\mathbb{R}_\theta^{2n})$ are elucidated.

Keywords: Space-Time Symmetries, Differential and Algebraic Geometry, Non-Commutative Geometry.

JHEP04(2002)026

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1. Introduction

Over five years ago, Connes gave the first axiomatics for first-quantized fermion fields on (compact) noncommutative varieties, the so-called spectral triples [1]. Shortly afterwards, it was realized that compactification of matrix models in M-theory leads to noncommutative tori [2]. The next logical step, introducing quantum fields in the framework of Connes' noncommutative spaces, was taken in [3]. Then it came the discovery by Seiberg and Witten that the dynamics of open strings, "rigidified" by the presence of a magnetic field, is described by a noncommutative geometry associated to the Moyal product [4]. Since then, field theories on Moyal-type spaces, including noncommutative tori, have been scrutinized extensively; good reviews are [5, 6].

The magnetic field breaks of course Lorentz invariance; a model which emphasizes full Poincarè invariance had been proposed in [7].

By now it is clear that (as it was bound to happen) Moyal-type spaces do *not* suffice to the needs of string and noncommutative field theories. Apart from the half-string product introduced long ago by Witten to construct a string field theory [8], there has been the appearance of "fuzzy sphere" products in the theory of strings and branes [9]. Moreover, the nonlocality of noncommutative gauge theories induces, at the level of the effective actions, generalizations of the Moyal product, including a ternary product [10]. These new products are most often either products of finite matrices (the fuzzy algebras) or can be reduced to the Moyal product — also for the Witten product [11]. On the formal side, a more ambitious attempt has been made in [12].

Very recently, there has been an effort [13] to face the shortfall, by introducing a non-formal star product on $\mathbb{R}^3$. (The Moyal product on $\mathbb{R}^3$ is trivial in the sense that it just extends the Moyal product on $\mathbb{R}^2$, with one direction remaining "totally commutative".) The construction relies on "projecting" to the latter, with the help of the Hopf fibration, a relative of the Moyal product on $\mathbb{R}^4$: to wit, the twisted product associated to normal ordering. In other words, it harks back to the Jordan-Schwinger map. The product in [13] is rather directly related to the fuzzy sphere.

The method by Hammou and coworkers in [13] is too dependent on specifics to be readily generalized. Perhaps the time has come to *upgrade* the fabrication of star products to the industrial stage. This is one theme of the present paper.

To construct viable generalizations it is required, at the outset, to have noncommutative geometries which are analytically controlled — as the Moyal product is. That is to say, formal deformations of classical manifolds, of which there are plenty [14], do not nearly foot the bill. We will explicitly construct analytically controlled products based on each and every three dimensional Lie algebra; as the Moyal product generalizes the Heisenberg relation $[x_i, x_j] = i\theta_{ij}$, our deformed algebras are based on the relations $[x_i, x_j] = ic_{ij}^k x_k$, with constant c's.

Meanwhile, in a more mathematical vein, new *compact* spectral triples, i.e., noncommutative compact spin geometries, have been introduced in [15]. Namely, Connes and Landi there explained what an even-dimensional noncommutative sphere is. Related to this, Várilly introduced noncommutative orthogonal groups [16]. See also [17] in this connection. Connes and Dubois-Violette [18] have dealt in depth with the odd-dimensional case and, more recently, Figueroa, Landi and Várilly [19] extend the constructions to cover grassmannians and other homogeneous spaces as well.[1]

The deformation, in the sense of Connes–Dubois-Violette, of a sphere $\mathbb{S}^n$ is obtained via an algebraic deformation of the noncompact manifold $\mathbb{R}^{n+1}$, plus a reduction to a quotient algebra by setting $r^2 = 1$, where r^2 is a central quadratic element in the deformed $\mathbb{R}^{n+1}$ algebra. The method would not seem to work on $\mathbb{R}^3$: the only spherical spectral triple in two dimensions is the ordinary sphere.

[1] Let us point out here that, in the continuous category, noncommutative spheres were first introduced by Matsumoto [20], and that some distinctions of principle need to be drawn between Connes–Landi and Connes–Dubois-Violette spheres; also, the question of eventual symmetries of the latter from deformations of orthogonal groups seems to be open.

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The definition of spectral triple, giving rise to noncommutative spin manifolds, in [1] had several strong restrictions and mathematical conditions (see section 6 below) destined to ensure that the commutative case realizes a fully algebraic description of a compact spin manifold. The framework needs enlargement, and in fact, postulates for riemannian, not necessarily spin, manifolds [21] and for semi-riemannian (for instance, with lorentzian signature) spectral triples [22] have been proposed. However, in all of these generalizations, compactness is retained. This is a drawback from the standpoint of noncommutative field theory.

As its second theme, and partly with the aim of fitting the new star products into a general analytic framework, this paper introduces *noncompact* noncommutative geometries. Going into this new territory, the landscape changes, and the relation of the different natural examples becomes intricate. It turns out that, *mutatis mutandis*, the whole "seven-axiom" apparatus by Connes, can be pushed through to the noncompact case. This covers the commutative bedrock and, we conjecture, the noncompact spaces associated to the Connes–Landi and Connes–Dubois-Violette spheres, insofar as they are spectral triples. However, the Moyal algebra case is *not* covered by the more direct generalization.

The difficulties cluster around the "dimension axioms", numbered 1 and 6 in our reckoning (see section 6). In the noncompact situation, two definitions of "classical dimension" part ways. These are: (i) the "metric dimension", based on the asymptotic growth of the spectrum of the resolvent of the Dirac operator D, and (ii) the "homological dimension", based on the Hochschild homology of the underlying algebra, which a priori has nothing to do with D. The Hochschild homology of the Moyal algebra vanishes in degrees greater than zero; and so in some sense the dimension of that noncommutative space is zero; this will be confirmed by the Dixmier trace check.

At the present stage of exploration of the vast noncommutative world, it would seem nevertheless unwise to rule out the Moyal product, which centrality is rather enhanced by the first part of our investigation, for the sake of axiomatics.

So, by relaxing the axioms, we introduce as well a second class of noncompact geometries, that we term *star triples*. A star triple is a kind of spectral triple in which: a) the noncommutative algebra is obtained from a deformation, i.e., it is given by a star product on a linear space of functions and/or distributions on a given manifold; b) the Dirac operator (is possibly deformed, but) remains an ordinary (pseudo-)differential operator on that original manifold; this can be used to establish the dimension. Almost all known spectral triples are of this type. The star triple set of postulates recovers the Moyal subcase.

Let us return to our first subject. The strategy for that can be summarized in a few words: we look for *subalgebras* of the Moyal algebra, that realize, in the spirit of Dirac, deformations of Poisson structures on $\mathbb{R}^n$. Before indicating how some Moyal subalgebras give rise to the star products, let us point out that the paradigm of noncommutative manifolds, the venerable noncommutative torus [23], can be as well rigorously proven to be, after all, a Moyal subalgebra. See our discussion in section 5.

Our approach asks for familiarity with Moyal algebra. When Seiberg and Witten related the dynamics of strings to the Moyal product, they were not entering mathematically virgin land. For more than fifty years, going back to [24], with the precedent of [25],

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that product has been used to perform quantum mechanical calculations on phase space. Among the tools we import to noncommutative field theory from quantum mechanics in phase space, the Wigner transform (6.3) does not seem to have been used before. We have collected some of these in a long appendix, for the convenience of the reader. We consider that it pays to be acquainted with the body of extant results in the literature of phase-space quantum mechanics. Otherwise, one risks to miss the more effective approaches, costly mistakes (more on that in section 6 and the appendix) or, in the best of cases, laboured arguments to rediscover trivialities.

The approaches used in recent constructions of noncommutative geometries in the sense of Connes [15, 16, 18, 19] have in common that noncommutative tori are exploited to the full: the basic idea is to find actions of torus groups $\mathcal{T}^l$ (where $l \geq 2$) on the spheres and the orthogonal groups, and to twist them. This way noncommutativity is "injected" in the manifold from within, so to speak. Here, as advertised, we take a different tack: to "project" it, in the noncompact case, from known noncommutative manifolds — in the occurrence, the standard Moyal algebras.

To carry on our programme, suitable maps $\pi : \mathbb{R}^{2n} \to \mathbb{R}^d$ are needed; a Poisson structure $\{.,.\}_P$ on $\mathbb{R}^d$ is assumed given. In fact, such maps π are the oldest game in town. They were introduced by Lie, under the name of *Funktionengruppen*, or "function groups" — see [26] and also [27]. A Funktionengruppe in the sense of Lie is a collection $\mathcal{F}$ of functions of the canonical variables (q_i, p_j) on $\mathbb{R}^{2n}$ such that:

- $\mathcal{F}$ is closed by functional composition, and generated by a finite number (say, d) of its elements.
- $\mathcal{F}$ closes to a subalgebra of $C^\infty(\mathbb{R}^{2n})$ under the Poisson bracket.

In local terms, the construction of Funktionengruppen is essentially equivalent to the problem of finding Poisson maps π from $\mathbb{R}^{2n}$ to (suitable subsets of) $\mathbb{R}^d$. Those we can call (linear) symplectic realizations of the Poisson structure on the latter. In fact, $\mathcal{F} := \pi^* C^\infty(\mathbb{R}^d)$ gives the function group.[2] But the original formulation by Lie is more in the spirit of noncommutative geometry.

This paper revolves around the observation that, under favourable circumstances, Lie's Funktionengruppen are also closed under the Moyal product.

For simplicity, in this connection, we concentrate on nontrivial deformations of $\mathbb{R}^3$ that can be derived from Moyal products on $\mathbb{R}^4$. Deformations of $\mathbb{R}^3$ are of interest for D3-brane field theories in which the brane's time direction is commuting; these are the relevant ones for the AdS/CFT correspondence. Also, they can be applied to world-volumes of euclidean D2-branes. Our scheme, nevertheless, does not suffer any limitation by dimension, and its wider applicability will be evident.

We begin, in section 2, by classifying all Poisson structures on $\mathbb{R}^3$. There is a wealth of them. A particularly attractive class of Funktionengruppen related to linear Poisson brackets is selected in section 3. The reader is advised to at least scan the appendix for

[2] Since the maps π^* are different for the several function groups considered in this paper, we should actually use the notation $\pi^*_{\mathcal{G}}$; we will however omit the subscript not to burden the notation.

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the notation, before going further. Section 4 is devoted to the proof of the main theorem. The corresponding star products are constructed in section 5. Then we examine properties of the resulting algebras case by case. Section 6 turns to noncompact noncommutative spin geometries and star triples. We indicate how a direct extension of Connes'axioms covers the basic (compact and) commutative noncompact cases, and also show that Moyal algebras are noncompact star triples in our sense. In section 7, we examine the standing of the new star products in regard to noncompact star triples postulates. Finally, in section 8, we summarize the results, and indicate some of the promising avenues thereby open.

2. Poisson structures on $\mathbb{R}^3$

We indicate the coordinates on $\mathbb{R}^4$ with a "phase space" notation: $\mathbb{R}^4 \ni u = (q_1, q_2, p_1, p_2)$. To proceed, we review the classification procedure of Poisson structures on $\mathbb{R}^3$, due to Grabowski, Perelomov and one of us [28].

The basic idea, to classify bivector fields Λ that could lead to Poisson structures on orientable manifolds, is to effect the inner product of Λ with the fundamental form. This allows to formulate (among other things) integrability conditions in terms of forms. On $\mathbb{R}^3$ it leads to a Casimir 1-form.

Consider coordinates provisionally called (x_1, x_2, x_3) on $\mathbb{R}^3$, and the fundamental form $\Omega = dx_1 \wedge dx_2 \wedge dx_3$. Let Λ denote the bivector field corresponding to any given Poisson bracket:

$$\Lambda = \sum_{i<j} c_{ij}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}\,; \tag{2.1}$$

or intrinsically $\Lambda(df \wedge dg) = \{f, g\}$. By contracting Λ with Ω we find

$$i_\Lambda \Omega = \frac{1}{2} \epsilon_{ijk} c_{ij}(x) dx_k \tag{2.2}$$

i.e.,

$$i_\Lambda \Omega = A_k(x) dx_k =: \alpha\,, \tag{2.3}$$

with $A_k = \frac{1}{2}\epsilon_{ijk} c_{ij}(x)$. The Jacobi identity for the Poisson brackets associated to Λ

$$\{x_i, x_j\} = c_{ij}(x)$$

is easily seen to be equivalent to the integrability condition for the Pfaff equation associated to α:

$$d\alpha \wedge \alpha = 0\,, \tag{2.4}$$

that is to say, in classical language

$$\vec{A} \cdot \operatorname{curl} \vec{A} = 0\,. \tag{2.5}$$

In conclusion, Poisson structures are characterized by 1-forms which admit an integrating factor; this characterization turns to be useful for our purposes. Locally we have $\alpha = f d\phi$ and

$$\{x_i, x_j\} = \epsilon_{ijk} f \frac{\partial \phi}{\partial x_k}.$$

The symplectic leaves for Λ are the level sets of ϕ, on which (the pull–back of) α vanishes. The 1-form α is what we call the *Casimir form* for the given brackets.

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From now on, we consider a restricted class of Poisson structures, for which the components of the bivector field Λ are linear in the coordinates (in particular we reluctantly refrain here from studying the quadratic case, related to Sklyanin-type algebras). That is, α has the particularly simple form $A_j(x) = M_{ij}x_i$, with M a constant matrix. This is necessary and sufficient to describe all the three dimensional Lie algebras over the reals.

A real finite-dimensional Lie algebra $\mathcal{G}$ with Lie bracket $[\cdot,\cdot]$ defines in a natural way a Poisson structure $\{\cdot,\cdot\}$, on the dual space $\mathcal{G}^*$ of $\mathcal{G}$. One is allowed to think of $\mathcal{G}$ as a subset of the ring of smooth functions $C^\infty(\mathcal{G}^*)$. Choosing a linear basis $\{E_i\}_1^n$ of $\mathcal{G}$, and identifying them with linear coordinate functions x_i on $\mathcal{G}^*$ by means of $x_i(x) = \langle x, E_i\rangle$ for all $x \in \mathcal{G}^*$, we define the fundamental brackets on $\mathcal{G}^*$ by the expression $\{x_i, x_j\}_\mathcal{G} = c_{ij}^k x_k$ where $[E_i, E_j] = c_{ij}^k E_k$ and c_{ij}^k denote the structure constants of the Lie algebra. The Poisson bracket $\{.,.\}_\mathcal{G}$ is associated to a bi-vector field Λ which is locally given by

$$\Lambda = \sum_{i<j} c_{ij}^k x_k \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}\,. \tag{2.6}$$

Here the $c_{ij}^k := \epsilon_{ijl}M_{kl}$ have all the required properties.

The classification of three dimensional Lie algebras goes back to Bianchi [29]; see also [30]. We present here such classification, which will give rise to inequivalent star products on $\mathbb{R}^3$, in terms of the one-form $\alpha = M_{ij}x_i dx_j$ [31, 32].

Decompose α into its symmetric and antisymmetric parts:

$$\alpha = \frac{1}{2}(M_{ij} - M_{ji})x_i\, dx_j + \frac{1}{2}(M_{ij} + M_{ji})x_i\, dx_j\,. \tag{2.7}$$

Defining $a_k = \epsilon_{ijk}M_{ij}$ we get

$$\alpha = \frac{1}{2}a_k\epsilon_{ijk}(x_i\, dx_j - x_j\, dx_i) + \frac{1}{2}\, d(M_{ij}x_i x_j). \tag{2.8}$$

Recall that the Jacobi identity implies $d\alpha \wedge \alpha = 0$. We have two cases: either the form α is closed (and therefore exact since we are in $\mathbb{R}^3$):

$$\alpha = \frac{1}{2}\, d\,(M_{ij}x_i x_j) \qquad i,j \in \{1,2,3\}. \tag{2.9}$$

Or there exists a vector field, say $X_1 = \partial/\partial x_1$, which is in the kernel of both α and $d\alpha$. In this case we have

$$\alpha = h(x_2\, dx_3 - x_3\, dx_2) + \frac{1}{2}\, d(M_{ab}x_a x_b) \qquad a,b \in \{2,3\}, \tag{2.10}$$

with the constant $h \neq 0$. We can bring both cases to normal form. The first by diagonalizing the symmetric part of M, while for the second we use a rotation matrix which preserves the form $x_2\, dx_3 - x_3\, dx_2$. We find therefore, using the notation $x = x_1, y = x_2$ and $w = x_3$:

A. For the case of α closed:

$$\alpha = \frac{1}{2}\, d\left(ax^2 + by^2 + cw^2\right).$$

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B. For the case of α not closed:

$$\alpha = \frac{1}{2}h(y\,dw - w\,dy) + \frac{1}{2}\,d\left(by^2 + cw^2\right).$$

In summary, any three dimensional Lie algebra is characterized by the Casimir form

$$\alpha = h(y\,dw - w\,dy) + \frac{1}{2}\,d\left(ax^2 + by^2 + cw^2\right) \tag{2.11}$$

when the real parameters h, a, b, c are appropriately selected. This yields the Poisson brackets

$$\{x, y\} = cw + hy, \qquad \{y, w\} = ax, \qquad \{w, x\} = by - hw \tag{2.12}$$

while the Jacobi identity

$$d\alpha \wedge \alpha = 2h\,a\,x\;dy \wedge dw \wedge dx = 0 \tag{2.13}$$

holds true if and only if $ha = 0$. Thus we have two essentially different classes of algebras: those corresponding to the closed Casimir form ($h = 0$, case A), and ($a = 0$, case B), those corresponding to the Casimir form (2.10).

We list the algebras, indicating their relation to Bianchi's [29] classification of three dimensional Lie algebras in nine families, according to the dimension of $\mathcal{G}'$, the derived algebra. Our approach gives rise to ten families.

In the first situation, when $\alpha = d\mathcal{C}$, let us call $\mathcal{C}$ a Casimir function. In case A the parameters a, b, c, when different from zero, may be all normalized to modulus one. This grouping includes six different isomorphism classes of Lie algebras:

A.1 su(2) $\simeq$ so(3) with a, b, c all different from 0 and of the same sign. A basis can be chosen so that $a = b = c = 1$ (similar remarks will be understood in what follows, when pertinent). This corresponds to type IX from Bianchi's classification.

A.2 e(2), the algebra of the euclidean group in two dimensions, which may be obtained by contraction from the previous class, say $a \to 0$. This corresponds to an algebra of type VII in Bianchi's classification, that we can term VII_0.

A.3 sl$(2, \mathbb{R})$ $\simeq$ su$(1, 1)$ $\simeq$ so$(2, 1)$, with a, b, c all different from 0 and of different sign. This corresponds to type $VIII$.

A.4 iso$(1, 1)$, the Poincaré algebra in two dimensions, which may be obtained by contraction from the previous algebra. This corresponds to a type VI algebra; let us call it VI_0.

A.5 h(1), the Heisenberg-Weyl algebra, with only one parameter different from zero, for example $c > 0$. It may be obtained by further contraction from both e(2) and iso$(1, 1)$. This is type II.

A.6 The abelian algebra with $a = b = c = 0$. This corresponds to type I.

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The second case, with α not exact, includes four families of Lie algebras, two of which are further subclassified by the real parameter bc/h^2, the transformations of α leaving invariant the ratio between the determinants of the symmetric and antisymmetric parts of the matrix M_{ij}. We recall that all algebras of type B have $a = 0$ because of the Jacobi identity. One has

B.1 $h = 1$, $b = c = 0$, that is $\mathrm{sb}(2, \mathbb{C})$, the Lie algebra of the group of 2×2 upper(lower) triangular complex matrices with unit determinant. This corresponds to type V.

B.2 $h = 1$, $b = 0$, $c = 1$; this yields Bianchi's IV.

B.3 $h \neq 0$, $b = 1$, $c = -1$. When $h = 1$, then $\dim \mathcal{G}' = 1$ and this does correspond to type III. All the others, for which $\dim \mathcal{G}' = 2$, are type VI; we denote them by VI_h.

B.4 $h \neq 0$, $b = c = 1$: this is type VII_h.

Cases B.3 and B.4 are one-parameter families, as it is impossible to put all parameters equal to one with a similarity transformation. We will refer collectively to the previous four classes of algebras as $\mathcal{G}_h$. Notice that the Bianchi classification mixes algebras with closed and not closed Casimir form. Indeed, types VI_0 and VII_0 correspond to closed one-forms, while VI_h and VII_h with $h \neq 0$ do not. Our classification exhausts all possible three dimensional algebras.

3. Funktionengruppen

Consider now $\mathbb{R}^4$ with the canonical symplectic structure given by the Poisson brackets

$$\{q_i, p_j\} = \delta_{ij},$$

associated to the symplectic form

$$\omega = dq_1 \wedge dp_1 + dq_2 \wedge dp_2.$$

We are ready to show realizations $\pi : \mathbb{R}^4 \to \mathcal{G}^* \equiv \mathbb{R}^3$. We express π through the change of variables π^* that pulls smooth functions on $\mathbb{R}^3$ back to smooth functions on $\mathbb{R}^4$; in other words, we give the realizations by means of contravariant arrows in the category of smooth functions and maps. All that one has to do is to find three independent functions f_1, f_2, f_3 on $\mathbb{R}^4$ whose corresponding canonical brackets[3] have the required form (2.12). The Poisson map π is *not* required to be onto, nor a submersion, that is to say, to arise from a regular foliation of $\mathbb{R}^3$.

Several π-maps were constructed in [32], under the name of (generalized) classical Jordan–Schwinger maps. We give just a set of possible realizations of the maps f, together with a Casimir function $\pi^*\mathcal{C}$.

Let us consider first the cases A, in which the form $\alpha = d\mathcal{C}$ is exact.

[3] The corresponding brackets on $\mathbb{R}^3$ are Kostant–Kirillov–Souriau brackets, and thus the images of the realizations are unions of orbits of the coadjoint action; this theory is well known and does not bear repetition here.

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A.1 su(2)

$$f_1 \equiv \pi^* x = \frac{1}{2}(q_1 q_2 + p_1 p_2)\,, \qquad f_2 \equiv \pi^* y = \frac{1}{2}(q_1 p_2 - q_2 p_1)\,,$$
$$f_3 \equiv \pi^* w = \frac{1}{4}\left(q_1^2 + p_1^2 - q_2^2 - p_2^2\right) \tag{3.1}$$

satisfying the relations

$$\{\pi^* x, \pi^* y\} = \pi^* w\,, \qquad \{\pi^* y, \pi^* w\} = \pi^* x\,,$$
$$\{\pi^* w, \pi^* x\} = \pi^* y. \tag{3.2}$$

The Casimir function $\frac{1}{2}(f_1^2 + f_2^2 + f_3^2)$ is given by $\pi^*\mathcal{C} = \frac{1}{32}(p_1^2 + q_1^2 + p_2^2 + q_2^2)^2$.

A.2 e(2)

$$f_1 \equiv \pi^* x = \frac{1}{2}\left(q_2^2 + p_2^2 - q_1^2 - p_1^2\right), \qquad f_2 \equiv \pi^* y = q_2 + p_1,$$
$$f_3 \equiv \pi^* w = -q_1 - p_2 \tag{3.3}$$

with

$$\{\pi^* x, \pi^* y\} = \pi^* w\,, \qquad \{\pi^* y, \pi^* w\} = 0\,,$$
$$\{\pi^* w, \pi^* x\} = \pi^* y\,. \tag{3.4}$$

The Casimir function $\frac{1}{2}(f_2^2 + f_3^2)$ is given by $\frac{1}{2}(p_1^2 + q_1^2 + p_2^2 + q_2^2 + 2p_1 q_2 + 2p_2 q_1)$.

A.3 sl$(2, \mathbb{R})$

$$\pi^* x = \frac{1}{4}\left(p_1^2 + q_1^2 + p_2^2 + q_2^2\right), \qquad \pi^* y = \frac{1}{4}\left(q_1^2 + q_2^2 - p_1^2 - p_2^2\right),$$
$$\pi^* w = \frac{1}{2}(p_1 q_1 + p_2 q_2) \tag{3.5}$$

with

$$\{\pi^* x, \pi^* y\} = -\pi^* w\,, \qquad \{\pi^* y, \pi^* w\} = \pi^* x\,,$$
$$\{\pi^* w, \pi^* x\} = -\pi^* y\,. \tag{3.6}$$

The Casimir function $\frac{1}{2}(f_1^2 - f_2^2 - f_3^2)$ yields $\frac{1}{8}(q_1 p_2 - q_2 p_1)^2$.

A.4 iso$(1, 1)$

$$\pi^* x = \frac{1}{2}\left(p_1^2 + p_2^2 - q_1^2 - q_2^2\right), \qquad \pi^* y = p_1 + q_2\,, \qquad \pi^* w = -p_2 - q_1 \tag{3.7}$$

with

$$\{\pi^* x, \pi^* y\} = \pi^* w\,, \qquad \{\pi^* y, \pi^* w\} = 0\,,$$
$$\{\pi^* w, \pi^* x\} = -\pi^* y\,. \tag{3.8}$$

and the Casimir function $\frac{1}{2}(-f_2^2 + f_3^2)$ is $\frac{1}{2}(q_1^2 - q_2^2 - p_1^2 + p_2^2 + 2q_1 p_2 - 2q_2 p_1)$.

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A.5 h(1)

$$\pi^* x = q_1 \qquad \pi^* y = p_1 q_2 \qquad \pi^* w = q_2 \tag{3.9}$$

satisfying

$$\{\pi^* x, \pi^* y\} = \pi^* w\,, \qquad \{\pi^* y, \pi^* w\} = 0\,, \qquad \{\pi^* w, \pi^* x\} = 0\,. \tag{3.10}$$

The Casimir function $\frac{1}{2}(f_3^2)$ is $\pi^* \mathcal{C} = \frac{1}{2} q_2^2$.

The trivial case of the abelian algebra, obtained by taking all the coefficients in (2.12) equal to zero, may be realized by many functions.

We pointed out already that the maps π are not onto, in general. We have an onto map in the su(2) case, but, for instance, for the $\mathrm{sl}(2, \mathbb{R})$ case, $\mathbb{R}^4$ is projected onto a solid cone.

B. The remaining cases, for which α is not closed, can be treated in a uniform way by giving the realization

$$\pi^* x = -h(q_1 p_1 + q_2 p_2) - c q_2 p_1 + b q_1 p_2\,, \qquad \pi^* y = q_1\,, \qquad \pi^* w = q_2 \tag{3.11}$$

satisfying the relations

$$\begin{aligned} \{\pi^* x, \pi^* y\} &= h \pi^* y + c \pi^* w\,; \qquad \{\pi^* y, \pi^* w\} = 0\,, \\ \{\pi^* w, \pi^* x\} &= b \pi^* y - h \pi^* w\,. \end{aligned} \tag{3.12}$$

In this case there is no Casimir function, as the form $\alpha = h(q_1 dq_2 - q_2 dq_1) + b q_1 dq_1 + c q_2 dq_2$ is not closed.

The reader will have noticed that all realizations chosen are by means of quadratic and linear functions of the canonical coordinates; of course, essentially equivalent ones may be obtained through linear canonical transformations. However, it should be pointed out that there are not the only possible ones: there exist many other inequivalent realizations by different classes of functions. For instance, there is for case B.1 the alternative realization:

$$\pi^* x = -p_1 - p_2\,, \qquad \pi^* y = e^{q_1}\,, \qquad \pi^* w = e^{q_2}\,. \tag{3.13}$$

This form can be obtained from the previous one with a (singular) canonical transformation:

$$\begin{aligned} p_i' &= p_i q_i \\ q_i' &= \log q_i. \end{aligned}$$

4. The main principle

We steer now to prove that novel noncommutative products can be obtained on $\mathbb{R}^3$ from the Moyal product in four dimensions, via the Poisson maps described in the previous section. The Moyal product is described in the appendix, where we also set notations and collect some other relevant material. In the next section explicit formulae for the $*_{\mathcal{G}}$ are exhibited.

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The formulae themselves attest in each case that the star product in $\mathbb{R}^4$ of functions of the π^*x, π^*y, π^*w variables depends only on the π^*x, π^*y, π^*w variables, and an easy induction argument shows this to be the case in all instances. However, the more mathematically minded readers may prefer an *a priori* argument. That we give here.

Consider the noncompact space $\mathbb{R}^3$ with coordinates x, y, w. We want to define a deformed product $*_{\mathcal{G}}$ of the algebra of functions of x, y and w, with the property:

$$[x, y]_{*_{\mathcal{G}}} = i\theta\{x, y\}_{\mathcal{G}}\,, \tag{4.1}$$

for $\mathcal{G}$ any of the Lie algebras obtained in section 2.

Let us assume that a nonzero vector field H exists such that

$$L_H \pi^* x = L_H \pi^* y = L_H \pi^* w = 0\,, \tag{4.2}$$

with L_H the Lie derivative in the direction of H. Let z be a convenient (local, if you wish) fourth coordinate on $\mathbb{R}^4$. It is impossible that $L_H z = 0$, too. This means that, given any function $F(x, y, w)$, we can characterize π^*F as a function on $\mathbb{R}^4$ by the fact that

$$L_H \pi^* F = 0\,. \tag{4.3}$$

The crucial condition to meet then is that, given $F(x, y, w), G(x, y, w)$, it obtain

$$L_H (\pi^* F \star_\theta \pi^* G) = 0\,, \tag{4.4}$$

where $\star_\theta$ denotes the Moyal product.[4]

In effect, fulfilling equation (4.4) ensures that the following procedure is well defined: to multiply two functions of x, y, w, lift them to four dimensions to obtain functions on $\mathbb{R}^4$ that can be multiplied with the four dimensional Moyal $\star_\theta$ product. The resulting function will still be in the kernel of H, and it is therefore possible to regard it as a function on the three dimensional space. In other words: when we $\star$-multiply two functions belonging to the Funktionengruppe, the result still belongs to the Funktionengruppe, and:

$$\pi^*(F *_{\mathcal{G}} G) = \pi^* F \star \pi^* G \tag{4.5}$$

defines $F *_{\mathcal{G}} G$.

Vector fields with property (4.2) do exist. Consider the vector field X defined by

$$i_X \omega = -\pi^* \alpha\,. \tag{4.6}$$

In cases A, this is just the hamiltonian vector field associated to the Casimir function $\pi^*\mathcal{C}$. (We leave aside the trivial case in which $X = 0$.) By construction (4.2) holds with $H = X$.

To prove (4.4), we consider several situations.

We dispose first of the case of the abelian Lie algebra. Then the star product becomes the ordinary product and then of course the Funktionengruppe closes.

[4] Often, when this is not too liable to confusion, the suffix θ is omitted from the notation, likewise we will often suppress the $\mathcal{G}$ from the $*_{\mathcal{G}}$.

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For the cases A with quadratic Casimir, since $L_X(\pi^*F) = \{\pi^*F, \pi^*\mathcal{C}\}$, and in view of equation (A.11) of the appendix, property (4.4) can be rephrased as

$$[\pi^*F, \pi^*\mathcal{C}]_\star = [\pi^*G, \pi^*\mathcal{C}]_\star = 0 \quad \text{implies} \quad [\pi^*F \star \pi^*G, \pi^*\mathcal{C}]_\star = 0,$$

which is obvious.

For the cases A with quartic Casimir, we can consider instead the hamiltonian vector field H associated to the square root of $\pi^*\mathcal{C}$, which has the same properties. The result follows from the same argument. We remark that the idea is already present in [13].

In some cases, that line of reasoning in terms of the Lie derivative with respect to the vector field H can be recast in pure Poisson algebra terms.

For instance, for the su(2) case, we assert: the Poisson subalgebra generated by the quadratic functions $\pi^*x, \pi^*y, \pi^*w, f_H$ (with f_H the hamiltonian function associated to H) is the *Poisson commutant* of $f_H = q_1^2 + q_2^2 + p_1^2 + p_2^2$. In effect, by virtue of the Lie algebra isomorphism discussed in the appendix, the problem reduces to the simple linear algebra exercise of finding the centralizer of the matrix J of (A.4) in $\mathrm{sp}(4;\mathbb{R})$. This centralizer is given by

$$B = \begin{pmatrix} 0 & a & b & c \\ -a & 0 & c & d \\ -b & -c & 0 & a \\ -c & -d & -a & 0 \end{pmatrix}.$$

By the way, it is isomorphic to u(2). The quadratic form ${}^tuBu$ then reproduces the functions $\pi^*x, \pi^*y, \pi^*w, f_H$. The claim is proved.

But such a commutant is bound to be an (involutive) *Moyal* subalgebra, as $\{f_H, F\} = \{f_H, G\} = 0$ imply $\{f_H, F \star_\theta G\} = 0$, in view of (A.11) again. As $q_1^2 + q_2^2 + p_1^2 + p_2^2$ is a function of π^*x, π^*y, π^*w itself, we are done. The latter kind of argument goes back to the work by Bayen et al. [33], who reintroduced the Moyal product in theoretical physics.

For the cases B, the proof is slightly more involved. To make it clearer, let us look at the simpler case B.1.

First we reconsider the previous reasoning. They are but an infinitesimal version of the fundamental fact that linear symplectic transformations are implemented by inner automorphisms of the Moyal algebra. The implementers of those automorphisms are exhibited in the appendix in a fully explicit way. Now, just observe that π^*x, π^*y, π^*w in case B.1 generate the commutant of $\arctan(q_1/q_2)$. While the canonical transformations generated by the latter function are not linear, they still are implemented by inner automorphisms of the Moyal algebra. This happens because they give rise to bicanonical maps in the sense of Amiet and Huguenin [34] for the Moyal product. The detailed treatment of the bigger class of implementors would take us too far afield, however, and is left for another paper.

The validity of the formulae in the next section is not restricted by the fact that the image of the "scaffolding map" π can eventually be less than all of $\mathbb{R}^3$. Associativity of the $*_{\mathcal{G}}$ and the involution are inherited from the Moyal product.

The star products can be seen to give realizations of the enveloping Hopf algebras $\mathrm{U}(\mathrm{su}(2))$, $\mathrm{U}(\mathrm{sl}(2,\mathbb{R}))\ldots$ At the abstract level, it has been clear for a long time that the

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"quantization" of a linear Poisson structure is given by the universal enveloping algebra of the corresponding Lie algebra [35]. No big deal, then, except that we now possess a concrete, not just formal, realization-cum-completion of those algebras, which hopefully will allow a fresh attack on the long-standing problem of the existence of (metric) noncommutative geometries over them.

5. The new star products

We now illustrate the new deformed products obtained from the Moyal product on $\mathbb{R}^4$ through the reduction maps π defined in section 3. Each algebra of functions on $\mathbb{R}^3$ identified by the reduction map is closed with respect to the relative star product. When relevant, we also discuss representations and other notable aspects of the resulting algebras. We have no pretense in this section to give an exhaustive treatment of these products, and limit ourselves to explicit expressions of the products, as well as some other aspects considered relevant.

As an initial remark, note the similarity, advertised in the Introduction, of our procedure with the transition from the Moyal product to the noncommutative tori $\mathbb{T}^n_\theta$; the latter is identified also as a subalgebra of the Moyal algebra, associated with the invariance under an action of the group $\mathbb{Z}^n$. We outline how this can rigorously be proved, and refer to [36] for a more detailed treatment.

The Gelfand C^*-algebra representing the real plane is the algebra $C_0(\mathbb{R}^2)$ of continuous functions on $\mathbb{R}^2$, vanishing at infinity. This can be compactified (unitized) in several ways. The *maximal* compactification is given by the algebra $C_b(\mathbb{R}^2)$ of continuous and bounded functions on $\mathbb{R}^2$; this coincides with the *multiplier* algebra [37, section 1.3]: $fg \in C_0(\mathbb{R}^2)$ for all $g \in C_0(\mathbb{R}^2)$ iff $f \in C_b(\mathbb{R}^2)$. For all practical purposes one needs to work in the smooth category, so one is led to consider the dense Schwartz $\mathcal{S}$ subalgebra of smooth functions rapidly vanishing at infinity, together with all their derivatives, and the dense subalgebra of C_b of bounded smooth functions, all of whose derivatives are bounded. The latter space is denoted $\mathcal{O}_0$ in distribution theory. It is a subspace of the space $\mathcal{O}_c$ of polynomially bounded smooth functions, together with all their derivatives, with the degree of the polynomial bound independent of the derivative.

Endow $\mathbb{R}^2$ with the Moyal product $\star$. It turns out [38] that $\mathcal{S} \star_\theta \mathcal{S} = \mathcal{S}$ for all θ and (with an appropriately extended definition of $\star$) that $\mathcal{O}_0 \star \mathcal{S} = \mathcal{S} \star \mathcal{O}_0 = \mathcal{S}$. Intuitively evident, and true, but a bit harder to prove, is that $\mathcal{O}_c \star \mathcal{O}_c \subset \mathcal{O}_c$ and $\mathcal{O}_0 \star \mathcal{O}_0 \subset \mathcal{O}_0$. A proof was given long ago by Figueroa[5] [39]. Therefore $\mathcal{O}_0$ is not only a compactification of the *commutative* plane, but also a compactification of the *Moyal* plane.

The Moyal star is invariant by translation. Therefore the product of two periodic functions with a fixed period is also periodic with the same period. The subalgebras of periodic elements in $\mathcal{O}_0$ coincide with the smooth noncommutative torus algebras. End of the remark.

As we argued in the previous section, the definition of the new products is given by

$$\pi^*(F *_{\mathcal{G}} G) = \pi^* F \star \pi^* G \,. \tag{5.1}$$

[5] A Fourier-transformed version of Figueroa's result has appeared recently in [40].

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The explicit formulae for the new products are easily computed by means of the expansion of the Moyal product (equation (A.10) in the appendix). We introduce the notation $f(x_i*)$ for functions on $\mathbb{R}^3$, meaning they have to be intended as expansions in $*$-powers of the arguments. An analogous notation is used for functions on $\mathbb{R}^4$, meaning that they have to be intended as an expansion in $\star$-powers of the arguments. The Moyal product defined in (A.10) can be then rewritten in the language of pseudodifferential operators under the form

$$f\left(q^j\star, p^j\star\right)\star g\left(q^j, p^j\right) = f\left(q^j + \frac{i\theta}{2}\frac{\partial}{\partial p^j}, p^j - \frac{i\theta}{2}\frac{\partial}{\partial q^j}\right) g\left(q^j, p^j\right). \tag{5.2}$$

5.1 su(2)

The Poisson subalgebra generated by the quadratic functions $\pi^* x_i$, in (3.1) is the commutant of the space of functions of f_H with $f_H = p_1^2 + q_1^2 + p_2^2 + q_2^2$, essentially the square root of the Casimir. By means of (A.10) and (5.2) we find:

$$x_j *_{\mathrm{su}(2)} f(x_i) = \left\{x_j - \frac{i\theta}{2}\epsilon_{jlm}x_l\partial_m - \frac{\theta^2}{8}\left[\left(1 + x_k\partial_k\right)\partial_j - \frac{1}{2}x_j\,\partial_k\partial_k\right]\right\} f(x_i). \tag{5.3}$$

From this, general formulae similar to (5.2) are obtained. And so, the Casimir function in the sense of enveloping algebras is

$$\mathcal{C}(x_i*) = \frac{1}{2}(x*x + y*y + w*w) = \frac{1}{2}\left(x^2 + y^2 + w^2 - \frac{3}{8}\theta^2\right). \tag{5.4}$$

The function f_H has been recognized as the hamiltonian of the two–dimensional harmonic oscillator. It is therefore natural to define the complex quantities

$$z_i = q_i + ip_i \tag{5.5}$$

with the Poisson brackets $\{z_i, \bar{z}_j\} = -2i\delta_{ij}$. In this case the four dimensional $\star$ product becomes the operator product of the functions of the well known $a_i, a_i^\dagger$ of the harmonic oscillator, with θ playing the role of $\hbar$; this realizes in fact the Jordan-Schwinger map. A presentation of the Lie algebra su(2) in terms of creation and annihilation operators, $a_i, a_i^\dagger,\ i \in \{1, 2\}$, is provided by

$$X_1 = \frac{i(X_+ + X_-)}{2}, \qquad X_2 = \frac{X_- - X_+}{2}, \qquad X_3 = iX_0\,, \tag{5.6}$$

with

$$X_+ = a_1^\dagger a_2\,, \qquad X_- = a_2^\dagger a_1\,, \qquad X_0 = \frac{1}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right). \tag{5.7}$$

We can have these functions act on the usual Hilbert space of the two dimensional harmonic oscillator, with basis the cartesian kets: $|n_1, n_2\rangle$. The generators act as:

$$\begin{aligned} X_+\,|n_1, n_2\rangle &= \theta\sqrt{(n_1+1)n_2}\,|n_1+1, n_2-1\rangle \\ X_-\,|n_1, n_2\rangle &= \theta\sqrt{(n_2+1)n_1}\,|n_1-1, n_2+1\rangle \\ X_0\,|n_1, n_2\rangle &= \theta(n_1 - n_2)\,|n_1, n_2\rangle\,; \end{aligned} \tag{5.8}$$

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notice that the "energy" n_1+n_2 does not change (a consequence of the form of the Casimir), so that is it natural to change the basis and choose a basis of eigenstates of the hamiltonian and the angular momentum. Define

$$\psi_{lm} \equiv |l+m, l-m\rangle \qquad |n_1, n_2\rangle \equiv \psi_{\frac{n_1+n_2}{2}, \frac{n_1-n_2}{2}} \tag{5.9}$$

with $l \geq 0,\ -l \leq m \leq l$. Then (5.8) becomes

$$\begin{aligned} X_\pm\, \psi_{l,m} &= \theta\sqrt{l(l+1) - m(m\pm 1)}\, \psi_{l,m\pm 1} \\ X_0\, \psi_{l,m} &= \theta l(l+1)\, \psi_{l,m}\,. \end{aligned} \tag{5.10}$$

For each value of l (integer or half integer) there is a representation of su(2). The algebra of functions of $\mathbb{R}^3$ therefore reduces to a set of finite dimensional algebras, receptacles for representations of su(2). This algebra can be given an interesting interpretation. Each reduced block is the algebra of a fuzzy sphere [41, 42], of radius $\theta\sqrt{l(l+1)}$ in the oscillator representation. Therefore the three dimensional space is "foliated" as a set of fuzzy spheres of increasing radius.

It is truly remarkable that, conversely, the *Moyal* product can be obtained from the fuzzy sphere of [42], by means of group contraction from SU(2) to the Heisenberg group [43].

We can give a geometric interpretation of the new star product. Note that, with the exception of the zero orbit, the orbits of the hamiltonian system associated to f_H are circles. Functions of (x, y, w) correspond here to functions of (q_1, q_2, p_1, p_2) that remain invariant on those orbits. We are thus identifying $\mathbb{R}^3$ to the foliation of $\mathbb{R}^4$ by those trajectories. The orbits are all parallel and rest on spheres in $\mathbb{R}^4$. One circle and only one passes through each point different from 0. The corresponding maps $S^3 \to S^2$ are Hopf fibrations.

Recall that a three dimensional product related to the fuzzy sphere has been introduced in [13]. Although similar, the product described in this subsection is not identical to the one introduced by Hammou and coworkers. For instance, one can check that, if $r = \sqrt{x^2+y^2+w^2}$, then $x_j *_{\mathrm{su}(2)} r = r *_{\mathrm{su}(2)} x_j$ equals simply rx_j, which is not the case for their product. The difference is attributable to the product in [13] being based on normal ordering, while we use Weyl ordering. Except for that, it can be regarded as a particular case of our construction.

5.2 e(2)

In this case the Casimir function is quadratic, hence the Poisson subalgebra generated by the quadratic functions $\pi^* x_i$ (3.3) is the Poisson commutant of $\pi^*\mathcal{C}$. The correspondent star product is found to be

$$\begin{aligned} x *_{\mathrm{e}(2)} f(x,y,w) &= \left\{ x - \frac{i\theta}{2}\left(y\partial_w - w\partial_y\right) - \frac{\theta^2}{8}\left[4\partial_x + 2x\partial_x^2 + w\partial_x\partial_w + y\partial_x\partial_y\right] \right\} f \\ y *_{\mathrm{e}(2)} f(x,y,w) &= \left\{ y - \frac{i\theta}{2} w\partial_x \right\} f \\ w *_{\mathrm{e}(2)} f(x,y,w) &= \left\{ w + \frac{i\theta}{2} y\partial_x \right\} f\,. \end{aligned} \tag{5.11}$$

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The Casimir function is

$$\mathcal{C}(x_i*) = \frac{1}{2}(y*y + w*w) = \frac{1}{2}\left(y^2 + w^2\right), \tag{5.12}$$

and it coincides with the ordinary Casimir. The representation theory can be obtained from that of su(2) through standard contraction techniques.

5.3 $\mathrm{sl}(2,\mathbb{R})$

As for the su(2) case, the Casimir is here quartic. Then the Poisson subalgebra generated by the functions $\pi^* x_i$ is the commutant of $f_H = q_1 p_2 - q_2 p_1$, the square root of the Casimir function. The star product induced by $\mathrm{sl}(2,\mathbb{R})$ through the maps (3.6) is:

$$x_j *_{\mathrm{sl}(2,\mathbb{R})} f(x_i) = \left\{ x_j - \frac{i\theta}{2} g_{jk} c_{klm} x_l \partial_m - \frac{\theta^2}{8}\left[(1 + x_l\partial_l) g_{jk}\partial_k - \frac{x_j}{2} g_{lm}\partial_l\partial_m\right]\right\} f \tag{5.13}$$

where $g_{ij} = diag(1,-1,-1)$ is the invariant metric in $\mathrm{sl}(2,\mathbb{R})$. It is formally identical to the one relative to su(2), substituting g_{ij} for δ_{ij}. The Casimir function is:

$$\mathcal{C}(x_i*) = \frac{1}{2}(x*x - y*y - w*w) = \frac{1}{2}\left(x^2 - y^2 - w^2 - \frac{3}{8}\theta^2\right). \tag{5.14}$$

The orbits of the angular momentum f_H are also circles, but the quotient map from $\mathbb{R}^4$ to the 3-manifold with boundary $\{(x,y,w) : x^2 - y^2 - w^2 = {f_H}^2 \geq 0\}$ is topologically trivial.

5.4 $\mathrm{iso}(1,1)$

This case is similar to the euclidean case. The Casimir is quadratic and the star product induced by the Poincaré algebra is

$$\begin{aligned}
x *_{\mathrm{iso}(1,1)} f(x,y,w) &= \left\{ x + \frac{i\theta}{2}(y\partial_w + w\partial_y) + \frac{\theta^2}{8}\left[4\partial_x + 2x\partial_x^2 + w\partial_x\partial_w + y\partial_x\partial_y\right]\right\} f \\
y *_{\mathrm{iso}(1,1)} f(x,y,w) &= \left\{ y - \frac{i\theta}{2} w\partial_x \right\} f \\
w *_{\mathrm{iso}(1,1)} f(x,y,w) &= \left\{ w - \frac{i\theta}{2} y\partial_x \right\} f\,.
\end{aligned} \tag{5.15}$$

The Casimir function is:

$$\mathcal{C}(x_i*) = \frac{1}{2}(-y*y + w*w) = \frac{1}{2}\left(-y^2 + w^2\right). \tag{5.16}$$

It is local and coincides with the ordinary Casimir.

5.5 $\mathrm{h}(1)$

We now discuss the case associated to the Heisenberg Lie algebra, whose map from $\mathbb{R}^4 \to \mathbb{R}^3$ is (3.9). In this map the quantity p_2 never appears, and therefore the algebra generated by these generators is highly reducible (not surprisingly given the essential unicity of the

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representations of the Heisenberg algebra), and evaluated at a particular w it reduces simply to a copy of the Moyal algebra. The star product is given by:

$$\begin{aligned} x *_{\mathrm{h}(1)} f(x,y,w) &= \left\{ x + \frac{i\theta}{2} w\partial_y \right\} f \\ y *_{\mathrm{h}(1)} f(x,y,w) &= \left\{ y - \frac{i\theta}{2} w\partial_x \right\} f \\ w *_{\mathrm{h}(1)} f(x,y,w) &= wf \,. \end{aligned}$$

The Casimir function

$$\mathcal{C}(x_i *) = \frac{1}{2}(w * w) = \frac{1}{2} w^2 \tag{5.17}$$

coincides with the ordinary Casimir. In [44], an explicit expression for the product of two arbitrary monomials is given.

The three dimensional space in this case is given by the space spanned by q_1, q_2 and p_1 , with the choice of (3.9). This is in turn foliated by the product of a line (q_2) and a set of Moyal planes spanned by $p_1 q_2$ and q_1. Note that the plane $w = 0$ has different properties, as it is well known from the theory of representations of the Heisenberg algebra.

5.6 B: the algebras $\mathcal{G}_h$

These algebras corresponding to the nonclosed Casimir form may be treated in a unified manner. The star product induced by the quadratic realization of $\mathcal{G}_h$ (3.12) is:

$$\begin{aligned} x *_{\mathcal{G}_h} f(x,y,w) &= \left\{ x + \frac{i\theta}{2} \left[h(y\partial_y + w\partial_w) + cw\partial y - by\partial w \right] - \right. \\ &\qquad \left. - \frac{\theta^2}{4}(h^2 + bc) \left[2\partial_x + x\partial_x^2 + w\partial_x\partial_w + y\partial_x\partial_y \right] \right\} f \\ y *_{\mathcal{G}_h} f(x,y,w) &= \left\{ y - \frac{i\theta}{2}(hy + cw)\partial_x \right\} f \\ w *_{\mathcal{G}_h} f(x,y,w) &= \left\{ w - \frac{i\theta}{2}(hw - by)\partial_x \right\} f \,. \end{aligned} \tag{5.18}$$

This star product is not equivalent to the one obtained from the non linear realization (3.13). Concentrate on the $\mathrm{sb}(2,\mathbb{C})$ case. Here $h = 1$, $b = c = 0$. The representation of this algebra on $L^2(\mathbb{R}^2)$ is reducible, but it is indecomposable, a trademark of triangular algebras. The Weyl map on $L^2(\mathbb{R}^2)$ of the generators (3.11) yields:

$$\hat{X} = -\frac{1}{2} \sum_{i=1,2} (\hat{p}_i \hat{q}_i + \hat{q}_i \hat{p}_i) \,, \qquad \hat{Y} = \hat{q}_1 \,, \qquad \hat{W} = \hat{q}_2 \,, \tag{5.19}$$

where the $\hat{}$ denotes usual quantum operators. Consider the (nonorthonormal) basis:

$$\psi_{nm} = q_1^n q_2^m e^{-(q_1^2 + q_2^2)} \tag{5.20}$$

and define $\mathcal{H}_{NM}$, the Hilbert subspaces spanned by ψ_{nm} with $n > N, m > M$. It is easy to see that these spaces are left invariant by any function of the operators $\hat{X}, \hat{Y}$ and $\hat{W}$

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for any choice of N and M. The representation is therefore reducible. It is not however possible to block–diagonalize it, and it is therefore indecomposable.

The presentation of the precedent star algebras by unitaries and relations is a straightforward (if somewhat tedious) task. See the discussion in the appendix; for further detail, the reader is advised to consult [45].

It is interesting to compare our products with the formal products introduced by Kontsevich [14]. Both are $\mathcal{G}$-covariant. One can see that higher order cochains contain exactly the same type of terms, but the weights are different. Very likely, the products are equivalent in the very general sense of [14]. Our products differ in principle from Kontsevich's in that they are non-formal; equivalence in the formal sense does not signify representations on Hilbert space (if any) are equivalent. Also, the operators involved in the asymptotic expansion in θ for our products are not necessarily (bi-)differential, they can be pseudodifferential.

It would also be interesting to know how the present star products are related to the foliation algebras introduced by Connes [47].

6. Noncompact star triples

In this section and the next we lay the foundations of a theory of noncompact spectral triples. Although the problems are mathematical, our approach will be guided by a more utilitarian view proper to physicists.

A *compact* spectral triple is a triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is a unital pre-C^*-algebra, $\mathcal{H}$ is a Hilbert space carrying a representation of $\mathcal{A}$ by bounded operators, and D is a selfadjoint operator on $\mathcal{A}$, with compact resolvent $R_D(\lambda) := (D - \lambda)^{-1}$, such that the commutator $[D, a]$ is also bounded on $\mathcal{H}$, for each $a \in \mathcal{A}$.

Spectral triples come in two parities, odd and even. In the odd case, there is nothing new; in the even case, there is a grading operator χ on $\mathcal{H}$ (a bounded selfadjoint operator satisfying $\chi^2 = 1$), such that the representation of $\mathcal{A}$ is even and the operator D is odd; thus each $[D, a]$ is a bounded odd operator on $\mathcal{H}$.

In the noncompact case we need to modify this definition, as $\mathcal{A}$ will no longer be unital, and, more to the point, the commutative example indicates that compactness of $R_D(\lambda)$ is out of question. The generalization was pointed out by Connes himself in [46]: we ask $aR_D(\lambda)$ to be compact for any $a \in \mathcal{A}$.

As hinted in the Introduction, noncommutative compact spin geometries are spectral triples satisfying several extra conditions, arising from consideration of the algebraic properties of ordinary spin and metric geometry. Seven such properties were put forward in [1]. Here we just summarize them; a more complete account is given in [37, section 10.5].

1. *Classical dimension*: There is a unique nonnegative integer n, the "classical dimension" of the geometry, for which the eigenvalue sums $\sigma_N := \sum_{0 \le k < N} \mu_k$ of the compact positive operator $|D|^{-n}$ satisfy $\sigma_N \sim C \log N$ as $N \to \infty$, with $0 < C < \infty$. The coefficient is written $C = \mathrm{Tr}^+ |D|^{-n}$, where Tr^+ denotes the Dixmier trace if $n \ge 1$; it coincides essentially with the Wodzicki residue [37, Chapter 7]. This n is

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even if and only if the spectral triple is even. (When $\mathcal{A} = C^\infty(M)$ and D is a Dirac operator, n equals the ordinary dimension of the spin manifold M).

2. *Regularity*: Not only are the operators a and $[D, a]$ bounded, but they lie in the smooth domain of the derivation $\delta(T) := [|D|, T]$. (When $\mathcal{A}$ is an algebra of functions and D is a Dirac operator, this smooth domain consists exactly of the C^∞ functions.)

3. *Reality*: There is an antiunitary operator C on $\mathcal{H}$, such that $[a, Cb^*C^{-1}] = 0$ for all $a, b \in \mathcal{A}$ (thus $b \mapsto Cb^*C^{-1}$ is a commuting representation on $\mathcal{H}$ of the "opposite algebra" $\mathcal{A}^\circ$, with the product reversed). Moreover, $C^2 = \pm 1$, $CD = \pm DC$, and $C\chi = \pm\chi C$ in the even case, where the signs depend only on $n \bmod 8$. (In the commutative case, C is the charge conjugation operator on spinors.)

4. *First order*: The bounded operators $[D, a]$ commute with the opposite algebra representation: $[[D, a], Cb^*C^{-1}] = 0$ for all $a, b \in \mathcal{A}$.

5. *Finiteness*: The algebra $\mathcal{A}$ is a pre-C^*-algebra, and the space of smooth vectors $\mathcal{H}_\infty := \bigcap_k Dom(D^k)$ is a finitely generated projective left $\mathcal{A}$-module. (In the commutative case, this yields the smooth spinors.)

6. *Orientation*: There is a *Hochschild n-cycle* c, on $\mathcal{A}$ with values in $\mathcal{A} \otimes \mathcal{A}^\circ$, whose natural representative is denoted $\pi_D(c)$. Such an n-cycle is usually a finite sum of terms like $(a \otimes b) \otimes a_1 \otimes \cdots \otimes a_n$ which map to operators

$$\pi_D((a \otimes b) \otimes a_1 \otimes \cdots \otimes a_n) := aCb^*C^{-1}\,[D, a_1] \cdots [D, a_n]\,, \tag{6.1}$$

and c is the algebraic expression of the *volume form* for the metric determined by D. This volume form must solve the equation

$$\pi_D(c) = \chi \quad \text{(even case)}, \qquad \text{or} \qquad \pi_D(c) = 1 \qquad \text{(odd case)}. \tag{6.2}$$

7. *Poincaré duality*: The index map of D determines a nondegenerate pairing on the K-theory of the algebra $\mathcal{A}$. This is related to the existence of Morita duality between $\mathcal{A}$ and the "quantum Clifford algebra" constructed from $\mathcal{A}$ and D [50]. (In the commutative case, the Chern homomorphism matches this nondegeneracy with Poincaré duality in de Rham (co)homology.)

For proofs of the fact that when $A = C^\infty(M)$ the usual apparatus of geometry on spin manifolds (spin structure, metric, Dirac operator) can be fully recovered from these seven conditions in the compact case, see [48] and [37, Chap. 11].

The question is now to modify these conditions for our needs, since we deal with noncompact manifolds. This we do guided more by physical insight than by pretensions of rigour; all examples we consider have the topology of euclidean spaces. In the Moyal case, whenever we need to fix ideas, and in consonance with the first part of the paper, we think of $\mathbb{R}^4_\theta$.

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The minimum requirement for new postulates, beyond being consistent with the axioms for compact spaces, is to cover the commutative case. In particular, we give ourselves just the task of checking whether the linear spaces $\mathbb{R}^n$ with their usual spin structure satisfy the conditions laid down below. The inverse task of recovering that spin structure from the axiom is to follow the pattern established in [48] and [37]. It is left for a better occasion. Related ideas have been discussed in [50]. We have already indicated that, when needed, we deal with *star triples*, defined as deformations of commutative spectral triples. The latter will be seen to cover as well the Moyal product cases. This keeps great heuristic value: as a consequence, the Chamseddine–Connes spectral action principle [49] can be extended to the Moyal framework [36].

For convenience, we restate the definition of noncompact spectral triple, and carry the discussion of the "strict" and the "star triple" framework in parallel.

We define: an odd *noncompact* spectral triple is a triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is a non-unital pre-C^*-algebra, $\mathcal{H}$ is a Hilbert space carrying a representation of $\mathcal{A}$ by bounded operators, and D is a selfadjoint operator on $\mathcal{A}$, such that $fR_D(\lambda)$ is compact and the commutator $[D, f]$ is also bounded on $\mathcal{H}$, for each $f \in \mathcal{A}$.

An odd *noncompact star triple* is a triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is a non-unital pre-C^*-algebra, given by a star product on some space of functions and/or distributions on a spin^c manifold (in other words, we consider also a commutative product on the vector space $\mathcal{A}$), $\mathcal{H}$ is a Hilbert space carrying a representation of $\mathcal{A}$ by bounded operators, and D is a selfadjoint pseudodifferential operator on the manifold, such that $fR_D(\lambda)$ is compact for each $f \in \mathcal{A}$, and all commutators $[D, f]$ are bounded on $\mathcal{H}$.

In the even case, a grading operator χ on $\mathcal{H}$ is added in the same way as before.

Thus the first hurdle is to prove compactness of the operators $fR_D(\lambda)$ — where the action of f by left star multiplication is understood. This typically is easy for star product algebras: they often give rise to compact — indeed trace class — operators.

Both in the commutative and the Moyal case, take for $\mathcal{A}$ the algebra $\mathcal{S}$ of Schwartz functions on $\mathbb{R}^{2n}$; for $\mathcal{H}$, a direct sum of copies of the space of square summable functions on $\mathbb{R}^{2n}$. More to the point, $\mathcal{H}$ is the space of spinors on $\mathbb{R}^{2n}$, topologically trivial. Finiteness and reality postulates prevent the Hilbert space on which $\mathcal{A}$ is represented from being "too small". We naturally want our spectral triples be irreducible in some sense; but it is important to realize that the $\mathcal{H}$ must carry, as well as the representation of $\mathcal{A}$, the one of $\mathcal{A}^\circ$ and a sundry list of operators.

The action of $\mathcal{A}$ by the Moyal star product on $\mathcal{H}$ is diagonal, so for analytical purposes we can assume only one copy of $L^2(\mathbb{R}^{2n})$ is present. The operators $Lf : g \mapsto f \star g$ are trace class, with

$$\operatorname{Tr} Lf = (2\pi\theta)^{-n} \int_{\mathbb{R}^{2n}} f \,.$$

Perhaps the simplest way to see that is to remember that the Wigner transform intertwines Lf with a multiple of the Schrödinger-like representation of $\mathcal{A}$ on $L^2(\mathbb{R})$ [51]. To be precise, there exists a linear map

$$W : \mathcal{S}(\mathbb{R}^{2n}) \to \mathcal{S}(\mathbb{R}^n) \otimes \mathcal{S}(\mathbb{R}^n)$$

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(unitary for an appropriate inner product) such that

$$W(Lf)W^{-1} = \pi_S(f) \otimes I\,,$$

where the action $\pi_S(f)$ is given by an integral kernel k_f related to f:

$$[\pi_S(f)\varphi](x) = \int_{\mathbb{R}^n} k_f(x,y)\varphi(y)\,dy\,.$$

In other words, $\pi_S(f)$ is the Weyl pseudodifferential operator associated to the "symbol" f. Moreover, $W(f \star g) = Wf \circ Wg$, where $\circ$ means composition of integral kernels. From the explicit form of π_S (see [37, sect. 3.5]):

$$k_f(x,y) = \frac{1}{(2\pi\theta)^n}\int_{\mathbb{R}^n} f\left(\frac{x+y}{2}, z\right) \mathrm{e}^{\,i(x-y)z/\theta}\,dz \tag{6.3}$$

it is clear that

$$\mathrm{Tr}\, Lf = \mathrm{Tr}\,\pi_S(f) = \frac{1}{(2\pi\theta)^n}\int_{\mathbb{R}^{2n}} f(q,p)\,dq\,dp < \infty\,.$$

Therefore, if $f \in \mathcal{S}$, then $fR_D(\lambda)$ is certainly compact.[6]

In the commutative limit, things are tougher, as neither f nor $R_D(\lambda)$ are compact. However, the theorems quoted in [52, chapter 4] save the day. These refer to operators (that appear naturally in scattering theory) of the form $f(u)g(-i\nabla)$, with ∇ the derivative operator corresponding to the coordinates u, and count among the more celebrated estimates in analysis.

Denote by F the Fourier transform. To be precise, the claim is that there exists a compact operator C such that, for any two spinors Ψ_1, Ψ_2, with $F\Psi_2$ in the domain of the multiplication operator g

$$\langle \Psi_1, C\Psi_2\rangle = \left\langle f\Psi_1, F^{-1}\left(gF\Psi_2\right)\right\rangle.$$

Now, for $g(D) = (D-\lambda)^{-1}$ on $\mathbb{R}^n$ we have $g \in L^{n+1}(\mathbb{R}^n)$. Also $f \in L^{n+1}(\mathbb{R}^n)$, obviously. Then $f(u)g(D)$ belongs to the Schatten class I^{n+1}. This is clear for $n = 1$, as we obtain a Hilbert–Schmidt operator; and for $n = \infty$. The general case follows by interpolation. Hence $f(u)g(D)$ is compact.

Now for the new postulates.

1. *Classical dimension*: On $\mathbb{R}^n$, and, more generally, on a geodesically complete spin manifold M with riemannian measure ν_g, the operator $f|D|^{-n}$ belongs to the Dixmier trace class for all $f \in \mathcal{A}$, the trace being nonzero in general. As the link with the Wodzicki residue *density* is kept (see below), we can compute integrals by means of the Dixmier trace Tr^+:

$$\int f(x)|\nu_g| = \frac{n(2\pi)^n}{\Omega_n}\,\mathrm{Tr}^+\left(f|D|^{-n}\right),$$

[6] By the way, the previous does *not* imply that $\pi_S(C_0)$ is the ideal of compact operators.

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where Ω_n denotes the volume of the sphere $\mathbb{S}^{n-1}$. Geodesical completeness is needed to make sure D is selfadjoint. This formula appears already in [37, Corollary 7.22].

The criterion fails spectacularly for the Moyal algebra case: because f is then trace class, its product with any power $|D|^{-k}$ has vanishing Dixmier trace! This can be understood as of the algebra $\mathcal{S}$ with the Moyal product having vanishing dimension somehow; such a precipitous "dimension drop" indicates a severe difficulty for the transfer of geometrical features from the commutative context, in the Moyal case. One could argue that $\mathcal{S}$ is too "small", when dealing with such product, and define $\mathcal{A}$ as the ideal of elements f such that Lf of the Dixmier class. For $n = 2$, the Moyal inverse of the harmonic oscillator hamiltonian (a gentle smooth function vanishing not too rapidly at infinity) provides an example. But such a procedure contains no information on the dimension![7]

A replacement is at hand in the context of star triples, in terms of the Wodzicki residue density. Let us just look at the Dirac operator, and embrace the "metric dimension" definition. As D is of commutative type, the only real trouble is that, as the manifold is not compact, the spectrum of D is continuous. Then we must use a criterion that makes no distinction between the discrete and the continuous spectrum cases. Such a criterion was proposed in [53] for (positive) elliptic pseudodifferential operators. Let A be one of such. The idea is to look at the *s*pectral density of A, formally written as $\delta(\lambda - A)$. This is an operator-valued distribution in $\mathcal{K}'$ (the space of distributions that possess momenta at all orders: see the definition of $\mathcal{K}$ a bit further on) with the property

$$A^n = \int \lambda^n \delta(\lambda - A).$$

Let $d_A(x, y; \lambda)$ denote the distributional kernel of $\delta(\lambda - A)$. For the coincidence limit of the kernel $d_{|D|}$ (with $|D|$ of order 1) as $\lambda \uparrow \infty$, in dimension n one has:

$$d(x, x; \lambda) \sim \frac{1}{(2\pi)^n} \operatorname{wres} |D|^{-n}(x)\lambda^{n-1} + \cdots, \tag{6.4}$$

where "wres" denotes the Wodzicki residue density.[8] This we take as our replacement criterion: there must exist a unique nonnegative integer n, the "classical dimension" of the triple, such that (6.4) holds. This is known to be valid [53] both for discrete and continuous spectra. In [53] it is also indicated what the subsequent coefficients are (in the Cesàro or average sense) and how to compute them.

[7] However, the dimension drop for the Moyal algebra is not too serious a problem, because the compactifications, i.e., all algebras interpolating between the one obtained by simply adding the multiples of 1 to the Schwartz space algebra, and the maximal multiplier algebra, can have nonzero Hochschild homology. (A. Connes, private communication).

[8] In case the reader gets nervous about the $|D|^{-n}$ notation, for the spectrum of D may well include zero, it can be replaced by $(D^2 + \epsilon)^{-n/2}$ in all cases. In other words, for simplicity we have assumed in the notation that D has a "mass gap" around zero; but this assumption we can easily dispense with.

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2. *Regularity*: There is no need to modify the second "axiom": the commutative case follows from regularity theorems for Sobolev spaces, just as in the compact case.

3. *Reality*: There is no need to modify the third "axiom". One takes for C the usual charge conjugation operator for spinors. One finds

$$Cf^* \star C^{-1}\Psi = \Psi \star f, \tag{6.5}$$

so indeed we have a commuting representation.

4. *First order*: There is no need to modify the fourth "axiom".

5. *Finiteness*: First of all, the algebra $\mathcal{S}$ is a pre-C^*-algebra, both for the ordinary and the Moyal product, and this is very easy to prove. We recall very briefly the argument from [37]. Let $(1+f)(1+g) = 1$, with $f \in \mathcal{S}$; then g must be in $\mathcal{O}_0$, and then in $\mathcal{S}$. The analogous property can be checked as well for the star product, using the sequence space decomposition introduced in [38].

   In order to deal with the rest of the finiteness requirement (and also for eventual consideration of the K-theory groups), we invoke unitizations (i.e., compactifications) $\tilde{\mathcal{A}}$ of $\mathcal{A}$. Consideration of the multiplier algebra $\mathcal{O}_M$ of polynomially bounded smooth functions, together with all their derivatives, and its subalgebras $\mathcal{O}_T$, in which the degree of the polynomial bound can go up only by one with each derivative, $\mathcal{O}_c$, already defined, and $\mathcal{K}$, in which the degree of the polynomial bound goes down by one with each derivative, impose themselves, as suitable compactifications of $\mathcal{S}(\mathbb{R}^4)$.

   Also, for us, the *Moyal algebra*, denoted $M(\mathbb{R}^4_\theta)$ or simply M_θ, is the *maximal* unitization of the Schwartz algebra $\mathcal{S}$ with the Moyal product. The precise definition is dealt with in the appendix, but we bring some material up here. Concerning the Moyal multiplier algebra there have been misleading statements in the literature recently as well. We welcome the occasion to clarify matters.

   Already we have asserted that, if f, g are Schwartz, $f \star_\theta g$ is also a Schwartz function. The product operation is *c*ontinuous and therefore the tracial identity

$$\int f \star_\theta g = \int fg,$$

   valid for any θ, allows the extension of the Moyal product via linear space duality [38], to large classes of distributions. Thus the expressions $T \star_\theta f, f \star_\theta T$, with T a tempered distribution, make sense; moreover, the multiplication is (separately) continuous on its two variables. Witness to the excellent smoothing properties of the Moyal operation, these products are always smooth. We say $T \in M_\theta$ if both $T \star_\theta f$ and $f \star_\theta T$ belong to $\mathcal{S}$, for all $f \in \mathcal{S}$.

   On the "size" of M_θ there have been conflicting claims. It certainly contains $\mathcal{K}$ and $\mathcal{O}_c$ as subalgebras [39], for all θ. But for instance, the authors of [40] seem to doubt that it is any bigger, whereas there have been claims that it includes *all* of tempered distributions! The truth is intermediate: whereas M_θ is large, containing in particular many distributions, it does not contain $\mathcal{O}_T$, and thus $\mathcal{O}_M$, the commutative multiplier

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algebra. The true net of algebras was described in detail in [38] and [51]. At the end of the appendix we exemplify a family of nice-looking, smooth, bounded functions in $\mathcal{O}_T$, to wit, exponentials with quadratic exponents. For the most part, they belong to M_θ, but there are exceptions. This will show: a) M_θ is bigger than $\mathcal{O}_c$; b) M_θ is smaller than $\mathcal{O}_T$, and thus than $\mathcal{S}'$; c) M_θ depends on θ.

We now postulate that the space of smooth vectors $\bigcap_k \mathrm{Dom}(D^k)$ is a finitely generated projective left $\tilde{\mathcal{A}}$-module, for some appropriate unitization $\tilde{\mathcal{A}}$. It is clear that the module of spinors foots the bill.

6. *Orientation*: We shall keep the Hochschild condition, but postulated for elements of $\tilde{\mathcal{A}}$ and $\tilde{\mathcal{A}} \otimes \tilde{\mathcal{A}}^\circ$. More precisely, we shall assume that it is possible to represent on $\mathcal{H}$ elements a of $\tilde{\mathcal{A}}$ such that the commutators appearing in the Hochschild equation are still bounded and the same equation (6.2) holds.

   It is really easy to see that this condition is verified, too, in the commutative case. The normalized volume form, say for $n = 4$, is

$$dx_1 \wedge \cdots \wedge dx_4,$$

   with $x_1 := q_1, \ldots, x_4 := p_2$. The corresponding Hochschild 4-chain in $\tilde{\mathcal{S}}(\mathbb{R}^4)$ is obtained by applying the skewsymmetrization operator

$$c := -\frac{1}{4!} \sum_{\sigma \in S_4} (-1)^\sigma x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes x_{\sigma(3)} \otimes x_{\sigma(4)}.$$

   This n-chain is patently a Hochschild cycle. We can represent it on $\mathcal{H}$, according to

$$\pi_D(c) := -\frac{1}{4!} \sum_{\sigma \in S_4} (-1)^\sigma \, [D, x_{\sigma(1)}] \cdots [D, x_{\sigma(4)}].$$

   Then $\pi_D(c) = \gamma_5 = \chi$.

   For the Moyal product this computation is identical as for the commutative one! But c in this case must be a Hochschild 4-boundary, so it cannot serve the essential purpose of representing the fundamental class of a noncommutative manifold.

   We are at a loss to replace the orientability axiom in the context of star triples. The formal computation can in principle always carried out, though — see the discussion in next section.

7. *Poincaré duality*: Let us just say that some nondegenerate pairing between the K-theory of $\mathcal{A}$ and its K-homology with compact supports must exist. That this happens in the commutative case follows from de Rham theory.

7. Noncompact star triples by deformation of $\mathbb{R}^3$

Beyond the commutative case, the Moyal one for star triples, and the noncompact spaces associated to the Connes–Landi and Connes–Dubois-Violette spheres, a natural question is whether the new star products give rise to noncompact noncommutative spin geometries, or at least to star triples in the sense examined in the last section.

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The analysis of the contents of this Pandora's box is still far off. However, some candidates to become "the" Dirac operators present themselves. Among the candidates having a sporting chance, one discerns at once operators of the form

$$D_\sigma = -i\,(\sigma_x\{x,.\} + \sigma_y\{y,.\} + \sigma_w\{w,.\})\,, \tag{7.1}$$

acting on $L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$, with $-i(\sigma_j\partial_j)$ substituted for them whenever x_j becomes central. These are actually differential operators, called sometimes hamiltonian and sometimes poissonian operators. Implicitly, we are using here the standard riemannian structure on $\mathbb{R}^3$. It is not quite clear whether the corresponding riemannian measure is appropriate for the needs of field theory.

An immediate, if apparently minor, difficulty presents itself: the coefficients of the partial derivatives in these differential operators are no longer constant, but linear functions of the coordinates. That means that ellipticity of D_σ is spoiled at the origin (and in some cases, at hyperplanes going through it), whereas the Wodzicki residue diverges at the same point(s). If we put aside this "ultraviolet divergence", we can see that postulate 1 for star triples is fulfilled for D_σ, in view of the general properties of elliptic operators [53].

Postulate 2 is inherited from the Moyal case. Postulate 3 can be guaranteed by construction, just as in the standard compact case. Note that (6.5) will still hold, for the $*_{\mathcal{G}}$ product. (We suppress $\mathcal{G}$ from the notation henceforth.) Postulate 4 can be satisfied in a similar way. All this means that, formally speaking, it is perfectly possible to define fermion fields on which our algebras act componentwise by the star product.

Postulates 5 (abstract first order) and 6 (orientability), represent equations for D which are not easy to fulfil in general; in particular (6.2) amounts to a difficult nonlinear equation. The strategy for fulfilling postulate 5 is transparent: if a tentative D is a quasiderivation, that is:

$$D(f * g) = \hat{D}f * g + f * Dg\,,$$

for some appropriate $\hat{D}$, then $[D, f]_*$ fulfils

$$[D, f]_* * g = \hat{D}f * g\,,$$

and thus it commutes with right $*$ multiplication. The operators D_σ of (7.1) are first order, in this abstract sense.

On the other hand, we have been generally unable to solve for (6.2) in a satisfactory way in all cases. This matters because it seems unlikely to us that these versions of the enveloping algebras have trivial Hochschild homology. We have checked, however, that with the operator D_σ it is possible to find a suitable formal cocycle for e(2), iso(1, 1), and sb$(2, \mathbb{C})$. For the Heisenberg algebra, in order to take into account the central element *and* to find the right dimensionality, we add as advertised the $-i\sigma_3\partial_w$ term; we are very close there to the Moyal case. It is not too difficult to show that, for example in the case e(2) the choice $a_0 = (12w^2y)^{-1}$, $a_1 = x, a_2 = y$ and $a_3 = x + w$, at the price of an a_0 singular on two hyperplanes, solves the equation:

$$\varepsilon_{ijk}\, a_0 * [D_\sigma, a_i]_* * [D_\sigma, a_j]_* * [D_\sigma, a_k]_* = 1\,. \tag{7.2}$$

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Similarly, other linear combinations for the x_i and a cubic inverse a_0 solve the other cases. In all cases the presence of a null $*$-commutator is crucial in the calculation. We have been unable to find a solution for the case of the semisimple algebras.

The different peculiarities of the products must have to do with Lie algebra cohomology; but at present we can just speculate. It might be that some enveloping algebras are not orientable in Connes' sense. This likely "fact of nature", as in the Moyal case, would not of course mean the corresponding new star products are physically uninteresting.

8. Conclusions

As we said in the Introduction, this paper has two main subjects. On one side there is the presentation of a new machinery to construct noncommutative spaces; on the other it deals with the issue of noncompact noncommutative differential geometries.

The method to construct deformed products we have introduced is based on the idea that a simpler product on a linear space can give rise under favourable circumstances, via a reduction process, to another product on a different manifold. This idea has tremendous potential. We have explored the reduction from four to three dimensions of a linear Poisson bracket corresponding to the Moyal product in four dimensions. But there is no obstruction in principle to generalize this machinery to the reduction from an arbitrary number of dimensions. In this case there will be many more algebras, with richer structures.

Another generalization can go along the lines of considering non (quadratic-)linear realizations of the linear Poisson brackets, or nonlinear brackets altogether. Yet another generalization can be to replace Lie algebras by some of their generalizations, such as Lie algebroids. And the list can go on. What we have done in this paper is to just play with the simplest instances of these products; but potentially there is a lot more to come.

The second theme of the paper stresses the issue of noncompactness of these geometries. Our motivations for this stress are mainly of physical origin, although they have bearing on mathematics; this is an important issue, even if the conclusions can only be tentative at the present stage. In the occurrence, we were faced with the task of generalizing the requirements spelled out by Connes for the compact case. We found this to be a feasible task, especially for (the reduction of spaces which are) deformations of commutative spaces. Hence the concept of star triple, for which the new products provided examples.

It is not clear that in all cases the orientability requirement, already in trouble for Moyal algebra, is satisfied for the putative Dirac operators. While one distinct possibility is that we have not been clever enough to find the right operator, we should not dismiss the possibility that orientability, as adapted from compact triples, is a requirement that may require further modification for the noncompact case.

A third strand running through the paper is the exploitation of the "nonperturbative" form of the Moyal product, and the construction of the multiplier algebra therefrom. Even if the Moyal algebra does not quite reach the lofty status of "noncommutative spin manifold", its analysis in depth pays dividends. In effect, the results of this paper point out to a kind of universal role of the Heisenberg group in noncommutative harmonic analysis.

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From the formal point of view much remains to be done. We have not dealt with the issue of irreducibility. Nor have we seriously dealt with the proper measure to use on the new noncommutative spaces, an important aspect if the aim is to have field theories on them. The measure is likely to be connected with the orientability issue, and so the study of the former may shed some light on the latter.

Acknowledgments

We thank A. El Gradechi, J. Grabowski, H. Grosse, R. Salmoni, A. Sitarz, R. Szabo, R. Wulkenhaar and A. Zampini for helpful discussions, A.P. Balachandran for stressing to us the relevance of the oscillator representation for the fuzzy sphere, and J.C. Várilly for carefully reading a preliminary version of the manuscript and many useful suggestions. The work of F.L., G.M. and P.V. was supported in part by the *Progetto di Ricerca di Interesse Nazionale* SInteSi 2000. J.M.G.-B., F.L. and P.V. like to thank the Erwin Schrödinger Institute in Vienna for hospitality during the course of this research.

A. Moyal product primer

This appendix recapitulates properties of Moyal multiplication relevant for our endeavour. Different (nondegenerate) Moyal products on $\mathbb{R}^4$ are in principle associated with an invertible antisymmetric matrix Θ_{ij} which, with a change of coordinates, can be expressed in the canonical form:

$$\Theta = \begin{pmatrix} 0 & 0 & -\theta_1 & 0 \\ 0 & 0 & 0 & -\theta_2 \\ \theta_1 & 0 & 0 & 0 \\ 0 & \theta_2 & 0 & 0 \end{pmatrix}, \tag{A.1}$$

with $\pm\theta_i$ the eigenvalues of Θ. A simple rescaling can then equate $\theta_1 = \theta_2 = \theta$; and for simplicity we have assumed the canonical Poisson bracket for the final coordinates. In this setting, the Moyal product $f \star_\theta g$ of two Schwartz functions f, g on $\mathbb{R}^4$ is defined by

$$f \star_\theta g(u) := \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} L^\theta(u, v, w)\, f(v) g(w)\, d\mu^\theta(v)\, d\mu^\theta(w), \tag{A.2}$$

where $u := (q, p)$; θ is a positive real parameter; $d\mu^\theta(v) := (\pi\theta)^{-4}\, d\mu(v)$ and the integral kernel L is given by

$$L^\theta(u, v, w) := \exp\left(\frac{2i}{\theta}(uJv + vJw + wJu)\right), \tag{A.3}$$

where J denotes the antisymmetric matrix:

$$J := \begin{pmatrix} 0 & \mathbb{I}_2 \\ -\mathbb{I}_2 & 0 \end{pmatrix}, g \tag{A.4}$$

with $\mathbb{I}_2$ the 2×2 identity matrix. Equivalent integral alternative formulae are

$$f \star_\theta g(u) := (\pi\theta)^{-4} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} f(u + s) g(u + t)\, e^{2isJt/\theta}\, ds\, dt\,, \tag{A.5}$$

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or

$$f \star_\theta g(u) := (2\pi)^{-4} \int_{\mathbb{R}^4} \int_{\mathbb{R}^4} f\left(u + \frac{1}{2}\theta J s\right) g(u+t)\, \mathrm{e}^{-ist}\, ds\, dt\,. \tag{A.6}$$

The popular Moyal series development is an *asymptotic* expansion of the previous in powers of θ [54]:

$$f \star_\theta g(u) \sim fg(u) + i\frac{\theta}{2}\{f,g\}(u) + \sum_{k=2}^{\infty} \left(i\frac{\theta}{2}\right)^k \frac{1}{k!}\, D_k(f,g)(u), \quad \text{as } \theta \to 0, \tag{A.7}$$

where the k-order bidifferential transvection operators

$$D_k(f,g)(q,p) = \frac{\partial^k f}{\partial q^k}\,\frac{\partial^k g}{\partial p^k} - \binom{k}{1}\frac{\partial^k f}{\partial q^{k-1}\partial p}\,\frac{\partial^k g}{\partial p^{k-1}\partial q} + \cdots + (-)^k \frac{\partial^k f}{\partial p^k}\,\frac{\partial^k g}{\partial q^k}$$

were also introduced by Lie [55] — in relation to aspects of classical invariant theory nowadays all but forgotten.

The development (A.7) becomes exact under conditions that are spelled out in [54]. Outside them, the integral form (A.2) or its siblings should be exclusively used. This is illustrated by the dramatically different nature of the divisors of zero for (A.2) and respectively for (A.7): the Moyal product of two functions whose supports do not meet in general is *not* zero. Also, "nonperturbative" properties like T-duality in the periodic case are almost evident from the correct, integral expressions.

From any of the previous formulae it follows that $(f \star_\theta g)^* = g^* \star_\theta f^*$, with f^* denoting complex conjugate of f. It is not difficult to verify that, as asserted at the end of section 5, $f \star_\theta g$ is also a Schwartz function. The product operation is *continuous* on $\mathcal{S}$; we refer the reader to [38], for all this and the key tracial identity

$$\int f \star_\theta g = \int g \star_\theta f = \int fg\,,$$

which is valid for any θ.

The last two remarks allow the extension of the Moyal product to large classes of distributions via linear space duality [38], by the formula

$$\langle T \star f, g\rangle := \langle T, f \star g\rangle\ ,$$

for T a tempered distribution (and similarly for the product with T from the right).

In fact, due to the excellent smoothing properties of the Moyal product, $T \star f$ is always a smooth function, although in general *not* of the Schwartz class. Now $M_L(\mathbb{R}^4_\theta)$, the left multiplier algebra, is defined as the subspace of tempered distributions that give rise to Schwartz functions when left multiplied by Schwartz functions; it takes no time to check that $M_L(\mathbb{R}^4_\theta)$ is indeed an algebra. The right multiplier algebra $M_R(\mathbb{R}^4_\theta)$ is analogously defined.

The unital *Moyal algebra* M_θ is then defined as $M_\theta := M_L(\mathbb{R}^4_\theta) \cap M_R(\mathbb{R}^4_\theta)$. The algebras M_θ for different θ are all different, but they are related by a simple dilation. In previous sections we referred to several interesting unital subalgebras of the Moyal algebra M_θ, and

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to the fact that $\mathcal{O}_M$, the space of polynomially bounded smooth functions, together with all their derivatives, is not one of them. M_θ is invariant by rescaled Fourier transform, to wit

$$FM_\theta = M_{4/\theta}\,;$$

therefore $\mathcal{O}'_M \in M_\theta$; in fact, the bigger $\mathcal{O}'_T \in M_\theta$. The dual space of distributions M'_θ is a dense ideal of the multiplier algebra M_θ, endowed with the natural locally convex topology. The M_θ are normal spaces of distributions, and all their derivations are inner.

We turn to the analysis of a relevant group of automorphisms of M_θ. Translations of $\mathbb{R}^4$ and real symplectic 4×4 matrices (defined by ${}^tSJS = J$) act on functions respectively by

$$\mathrm{sf}(u) := f(u-s); \quad \mathrm{Sf}(u) := f\left(S^{-1}u\right). \tag{A.8}$$

Let (s,S) denote an element of the inhomogeneous symplectic group $ISp(4;\mathbb{R})$, i.e., the semidirect product of the group of translations and the symplectic group, with group law

$$(s_1,S_1)(s_2,S_2) = \left(S_2^{-1}s_1 + s_2, S_1S_2\right).$$

As an immediate consequence of (A.2) and (A.8) we have equivariance of the twisted product:

$$(s,S)f \star_\theta (s,S)g = (s,S)(f\star_\theta g)\,. \tag{A.9}$$

We shall verify presently that the (s,S) are *inner* automorphisms. This is clear for the $(s,\mathbb{I})$ transformations, which correspond to ordinary exponentials $\exp(-isu)$, that is to say, they generate by the Moyal star product the Weyl algebra. So we concentrate on the $(0,S)$ transformations.

The Lie algebra $\mathrm{sp}(4;\mathbb{R})$ of infinitesimal symplectic transformations is formed by matrices L such that

$$JL + {}^tLJ = 0\,.$$

Linear and quadratic functions double as hamiltonians respectively for translations and linear symplectomorphisms; they generate infinitesimal linear inhomogeneous symplectic transformations. To be precise, note first that the matrix B is symmetric iff JB is infinitesimally symplectic. Let then $h_{(b,B)} = \frac{1}{2}\,{}^tuBu + {}^tbu$, where B is symmetric. Then $(b,L) \mapsto h_{(b,-JL)}$ is an isomorphism of Lie algebras between $\mathrm{isp}(4;\mathbb{R})$, with an obvious notation, and the Poisson subalgebra of quadratic-linear functions on $\mathbb{R}^4$; in particular $\{h_A, h_B\}$ vanishes iff $[JA, JB] = 0$.

It is useful to have explicit formulae for Moyal star products in which one of the factors h is a function of this kind. Using the asymptotic development of the Moyal product, here exact, one easily obtains [45]:

$$h\star_\theta f(u) = hf(u) + \frac{i\theta}{2}\,{}^t(Bu+b)J\,\mathrm{grad}\,f(u) + \frac{\theta^2}{8}\,\mathrm{Tr}\left[BJ\,\mathrm{Hess}\,f(u)J\right]. \tag{A.10}$$

It follows either from (A.9) or (A.10) that

$$[h,f]_{\star_\theta} = i\theta\,\{h,f\}\,, \tag{A.11}$$

i.e., the Moyal bracket and the Poisson bracket in the aforementioned case essentially coincide.

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The linear symplectic group Sp is not simply connected, and it is pertinent to consider its twofold covering, the so called metaplectic group Mp. Just as in the spin representation, to each element $A \in Mp$ corresponds a symplectic matrix $S(A)$, with $S(A) = S(-A)$. The elements of Mp are realized by unitaries Ξ_S, belonging to the multiplier Moyal algebra M, and defined up to a sign, such that

$$\Xi_S \star f \star \Xi_S^* = Sf\,, \tag{A.12}$$

for all f. Explicitly,

$$\Xi_S(u) = e^{i\alpha}\frac{4}{\sqrt{\det(\mathbb{I}_4+S)}}\exp\left(-{}^t u\,\frac{iJ(\mathbb{I}_4-S)}{\theta(\mathbb{I}_4+S)}\,u\right). \tag{A.13}$$

Note that $S \mapsto (\mathbb{I}_4-S)(\mathbb{I}_4+S)^{-1}$ is the Cayley transform, sending a symplectic matrix into an infinitesimally symplectic one: the matrix in the exponent must be symmetric. After many years of undeserved neglect, this important "nonlinear" relation between a Lie group and its Lie algebra has received an authoritative study in [56]. The Ξ_S are elements of the Moyal algebra M_θ not belonging to $\mathcal{O}_c$. The phase factor in (A.13) reflects the ambiguity inherent to (A.12), which, as indicated, can be reduced to a sign, so that

$$\Xi_S \star \Xi_{S'} = \pm\Xi_{SS'}\,. \tag{A.14}$$

The curious reader can directly check this formula with the help of the method of the stationary phase [45].

Of course, the ambiguity can be eliminated completely for uniparametric subgroups. Equations (A.13) and (A.14) are the starting point for a presentation of our algebras in terms of unitaries and relations. One has $\Xi_{M^{-1}SM}(u) = \Xi_S(Mu)$, where M is a symplectic matrix (which under some circumstances can be taken complex). This simplifies the calculation of Ξ_S, as it suffices to carry it out for "normal forms", representative of the orbits of the symplectic group under the adjoint action. Consider $S(\beta) = \mathbb{I}_4\cos\beta + J\sin\beta \in \mathrm{Sp}(4;\mathbb{R})$. Then

$$\Xi_{S(\beta)} = \sec^2(\beta/2)\exp\left(-{}^t u\frac{i}{\theta}\tan\frac{\beta}{2}u\right). \tag{A.15}$$

Now consider, for instance, the product associated to su(2). All the generators are *elliptic*, which means that $\det B > 0$, for the associated symmetric matrix. Therefore

$$e^{\pm itx_j\star} = \sec^2(t/2)\exp\left(\pm\frac{2ix_j}{\theta}\tan\frac{t}{2}\right).$$

The product associated to $\mathrm{sl}(2,\mathbb{R})$ contains hyperbolic elements ($\det B < 0$) as well; for them, hyperbolic functions replace the trigonometric ones in the formula. Parabolic elements are considered in [45]. Relations can be obtained simply by using (A.14).

For "exceptional" symplectic matrices with $\det(\mathbb{I}_4 + S) = 0$, formula (A.13) appears not to be well defined. In fact, we only need to tackle the case $S = -\mathbb{I}$, corresponding to $\beta = \pm\pi$ in the example, and the cases

$$S_1 = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix}; \qquad S_2 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

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One goes to the distributional limit of the previous expression (A.15), obtaining

$$\Xi_{S(\pm\pi)}(u) = \mp i\pi^2\theta^2\delta(u)\,;$$

for the other cases, a factor $\pi\theta\delta^{(2)}(Pu)$, is similarly obtained, where P projects over the corresponding two dimensional symplectic subspace. Notice that $\Xi_{S(\pm 2\pi)} = -1$, in spite of the fact that $S(2\pi) = 1$: the twofold covering is unavoidable. For other exceptional elements, it is enough [45] to factor out S_1 or S_2 or both, and use the product law (A.14).

Consider now the hyperbolic uniparametric subgroup $S(\alpha) = \mathbb{I}_4 \cosh\alpha + A \sinh\alpha \in \mathrm{Sp}(4;\mathbb{R})$, where

$$A = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

Then

$$M\left(\mathbb{R}^4_\theta\right) \ni \Xi_{S(\alpha)} = \mathrm{sech}^2(\alpha/2) \exp\left(-\frac{i\,2pq\,\tanh(\alpha/2)}{\theta}\right). \tag{A.16}$$

It is plain that the value $e^{-i\,2pq/\theta}$ is never reached. For good reason: the functions $e^{\mp i\,2pq/\theta}$ do *not* belong to the multiplier algebra $M(\mathbb{R}^4_\theta)$. Note the dependence on θ.

More generally, $e^{\pm i\,{}^t u G u/\theta}$ belongs to $M(\mathbb{R}^4_\theta)$ iff $-JG$ is in the range of the Cayley map; matrices with determinant -1 cannot be in that range. These are the "counterexamples" promised in section 6. The Wigner transform can be understood as an operator from $\mathcal{S}'(\mathbb{R}^{2n})$ to $\mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))$ by means of Schwartz's kernel theorem. It sends $M_L(\mathbb{R}^{2n}_\theta)$ isomorphically onto $\mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}(\mathbb{R}^n))$. In addition to this fact, one proof that $e^{\pm i\,{}^t u G u/\theta}$ does not belong to $M(\mathbb{R}^4_\theta)$ when $-JG$ is not in the range of the Cayley map uses only Fourier analysis.

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